The Kolmogorov-Obukhov Exponent in the Inertial Range of Turbulence: A Reexamination of Experimental Data[1]

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Abstract. In recent papers Benzi et al. presented experimental data and an analysis to the effect that the well-known "2/3" Kolmogorov-Obukhov exponent in the inertial range of local structure in turbulence should be corrected by a small but definitely non-zero amount. We reexamine the very same data and show that this conclusion is unjustified. The data are in fact consistent with incomplete similarity in the inertial range, and with an exponent that depends on the Reynolds number and tends to 2/3 in the limit of vanishing viscosity. If further data confirm this conclusion, the understanding of local structure would be profoundly affected.

1. Introduction

In 1941 Kolmogorov derived his famous scaling relations for the local structure of turbulence [16]; in particular he deduced that the second-order structure function in the inertial range was proportional to the $r^{2/3}$ where r is the separation of the points (For

[1]Supported in part by the Applied Mathematical Sciences subprogram of the Office of Energy Research, U.S. Department of Energy, under contract DE–AC03–76–SF00098, and in part by the National Science Foundation under grants DMS94–14631 and DMS89–19074.

definitions and analysis, see below). Obukhov [18] simultaneously determined the energy spectrum in the inertial range by similar means. Soon afterwards, Landau suggested that the Kolmogorov-Obukhov 2/3 (–5/3) exponent may be modified by the presence of intermittency, and various proposals for Reynolds-number-independent modifications have been made since then (see e.g. [17]).

On the other hand, the Kolmogorov-Obukhov scaling argument has been reexamined through the prism of modern scaling theory [4,6,7,9] which produced a Reynolds-number-dependent exponent in the structure function, tending to 2/3 in the limit of vanishing viscosity. This argument is supported by the near-equilibrium statistical theory of turbulence [13,14,15], as well as by vanishing-viscosity asymptotics [4,5,6,7,14].

The Kolmogorov-Obukhov theory was not derived from first principles such as as the Navier-Stokes equations, and contains additional assumptions which are open to debate. In the absence of general analytical solutions of the Navier-Stokes equations and of adequate computational data, the only way to check the theory is to subject it to experimental verification. Several experimentalists have claimed to have observed a correction to the 2/3 exponent, and that it was independent of Re; among the influential papers in this direction are the papers of Benzi et al. [11,12]. In [11] (page 389), the authors state that "the exponents. . . are the same in all experiments" (i.e., they are independent of Reynolds number), and the exponent in the second order structure function is "close but definitely different from the value 2/3 used by Obukhov". Our goal here is to refute this statement: to the extent that the data exhibited in [11] can be relied upon, they militate in favor of a Reynolds-number dependent exponent with a 2/3 vanishing-viscosity limit. The difference between the conclusions reached in [11] and the conclusions reached below is apparently due to the data not having been carefully enough examined in [11] for possible dependence on Re.

In the next section we provide a brief summary of scaling arguments as they apply to the structure functions in fully developed turbulence. We then present our analysis of the data and draw a conclusion.

2. Scaling in the local structure of turbulence

The quantities of interest in the local structure of turbulence are the moments of the

relative velocity field, in particular the second order tensor with components

$$D_{ij} = \langle (\Delta_{\mathbf{r}})_i (\Delta_{\mathbf{r}})_j \rangle, \tag{2.1}$$

where $\Delta_{\mathbf{r}} = \mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x})$ is a velocity difference between $\mathbf{x}$ and $\mathbf{x}+\mathbf{r}$. In incompressible flow that is in addition locally isotropic, all the components of this tensor are determined if one knows $D_{LL} = \langle [u_L(\mathbf{x}+\mathbf{r}) - u_L(\mathbf{x})]^2 \rangle$ where u_L is the velocity component along the vector $\mathbf{r}$.

To derive an expression for D_{LL} assume, following Kolmogorov, that for $r = |\mathbf{r}|$ small, it depends on $\langle \varepsilon \rangle$, the mean rate of energy dissipation per unit volume, r, the distance between the points at which the velocity is measured, a length scale Λ, for example the Taylor macroscale Λ_T, and the kinematic viscosity ν :

$$D_{LL}(r) = f(\langle \varepsilon \rangle, r, \Lambda_T, \nu), \tag{2.2}$$

where the function f is assumed to be the same for all developed turbulent flows. Introduce the Kolmogorov scale Λ_K, which marks the lower bound of the "inertial" range of scales in which energy dissipation is negligible:

$$\Lambda_K = \frac{\nu^{3/4}}{\langle \varepsilon \rangle^{1/4}}; \tag{2.3}$$

Clearly, the appropriate velocity scale is

$$u = (\langle \varepsilon \rangle \Lambda_T)^{1/3}, \tag{2.4}$$

and this yields a Reynolds number

$$Re = \frac{(\langle \varepsilon \rangle \Lambda_T)^{1/3} \Lambda_T}{\nu} = \frac{\langle \varepsilon \rangle^{1/3} \Lambda_T^{4/3}}{\nu} = \left(\frac{\Lambda_T}{\Lambda_K} \right)^{4/3}. \tag{2.5}$$

Dimensional analysis yields the scaling law:

$$D_{LL} = (\langle \varepsilon \rangle r)^{\frac{2}{3}} \ \Phi \left(\frac{r}{\Lambda_K} \ , Re \right) \ , \tag{2.6}$$

where as before, the function Φ is an unknown dimensionless function of its arguments, which have been chosen so that in the inertial range they are both large.

If one now subjects (2.6) to an assumption of complete similarity in both its arguments (i.e., one assumes that Φ tends to a finite non-zero limit when its arguments tend to infinity, see [1]), one obtains the classical Kolmogorov 2/3 law [16]

$$D_{LL} = A_0 (\langle \varepsilon \rangle r)^{\frac{2}{3}} \ , \tag{2.7}$$

from which the Kolmogorov-Obukhov "5/3" spectrum [18] can be obtained via Fourier transform. If one makes the assumption of incomplete similarity in r/Λ_K and no similarity in Re, (i.e., one assumes that as $(r/\Lambda_K) \to 0$ the function Φ has power-type asymptotics with Re-dependent parameters), as we have shown to be appropriate in certain shear flows [7,8], the result is

$$\frac{D_{LL}(r)}{(\langle\varepsilon\rangle r)^{2/3}} = C(Re)\left(\frac{r}{\Lambda_K}\right)^{\alpha(Re)}, \tag{2.8}$$

where C, α are functions of Re only. Expand C and α in powers of $\frac{1}{\ln Re}$, as is suggested by vanishing viscosity asymptotics, (for another example of this expansion, see [5,7]), and keep the two leading terms; this yields

$$D_{LL} = (\langle\varepsilon\rangle r)^{2/3}\left(C_0 + \frac{C_1}{\ln Re}\right)\left(\frac{r}{\Lambda_K}\right)^{\alpha_1/\ln Re}, \tag{2.9}$$

where C_0, C_1, α_1 are constants and the zero-order term in the exponent has been set equal to zero so that D_{LL} has a finite limit as $\nu \to 0$ [7]. According to (2.9), the exponent in D_{LL} is $Re-$dependent and converges to 2/3 as $Re \to \infty$. Note that the prefactor, (the "Kolmogorov constant"), is also $Re-$dependent, as has indeed been observed experimentally [19,20].

A further possibility is to subject D_{LL} to an assumption of complete similarity in Re and incomplete similarity in r/Λ_K, opening the door to Re-independent corrections to the 2/3 power. This possibility, discussed in [7], is incompatible with the existence of a well-behaved vanishing-viscosity limit for the second-order structure function, in contradiction with the theory in [5,6,7,14]. This assumption corresponds to the "extended similarity" discussed in [11,12].

The conclusion that the classical Kolmogorov-Obukhov value is obtained in the limit $Re \to \infty$ was reached in [13,14,15] by a statistical mechanics argument. Furthermore, the usual explanation for possible departures of the exponent from the Kolmogorov-Obukhov value is the need to account for intermittency. However, it was shown in [13,15] that the Kolmogorov-Obukhov value already takes intermittency into account; indeed, mean-field theories presented in [14] give exponents which differ from 2/3, and of course depend on the additional assumptions used to define a system to which a mean-field theory can be applied. In [7] it was argued that the 2/3 value corresponds to "perfect" intermittency, with the Re-dependent correction being created by the decrease in intermittency due to

viscosity. This physical picture is mirrored by the fact that we obtained the 2/3 exponent not by an assumption of complete similarity but as the vanishing-viscosity limit of a power-law derived from an assumption of incomplete similarity; for a more detailed explanation in a related problem, see [8].

Kolmogorov [16] proposed similarity relations also for the higher-order structure functions:

$$D_{LL\ldots L}(r) = \langle [u_L(\mathbf{x}+\mathbf{r}) - u_L(\mathbf{x})]^p \rangle,$$

where $LL\ldots L$ denotes L repeated p times; the scaling gives $D_{LL\ldots L} = C_p(\langle\varepsilon\rangle r)^{p/3}$. As is well-known, for $p = 3$ the Kolmogorov scaling is valid with no corrections. We shall not be concerned here with higher-order structure functions, for which the validity of the vanishing-viscosity arguments is at present unknown , and whose very existence in the limit of vanishing viscosity is doubtful [7].

3. A reexamination of the data of Benzi et al.

Our starting point is the graph in Figure 3 of the paper [11], which contains a plot of the second order structure function $\log D_{LL}$ as a function of the logarithm of the third moment D_{LLL}, whose dependence on r in the inertial interval is well-known to obey the appropriate Kolmogorov scaling and thus be proportional to r. This way of processing the data provides a longer interval in which the exponent can be seen, and also produces as an artifact a slope of 2/3 for separations r in the dissipation range, as is indeed carefully explained in [11]. The data come from four series of experiments in a small wind-tunnel: experiments labeled J in which the turbulence was produced by a jet at $Re = 300,000$ (based on the integral scale), experiments labeled C6 where the turbulence was produced by a cylinder at $Re = 6000$, experiments labeled C18, with a cylinder and $Re = 18000$, and experiments in which the turbulence was produced by a grid and Re was not specified in the paper; we have found from referees that the Reynolds number of the grid data was low, with no precise value. The various experimental procedures are detailed in [11] and we do not query them in any way.

The resulting values of $\log D_{LL}$ as a function of $\log D_{LLL}$ were plotted in Figure 3 of [11] without regard to Re, and a line was fitted to them as if they came from a single experiment. That line had slope roughly equal to 0.7, and this is the basis for the claim by Benzi et al. that the exponent is definitely larger than 2/3. However, it is obvious even

to the naked eye that the points that come from experiments with differing Re do not lign up well on that single line. We now show this lack of alignment in detail.

To our regret, Benzi et al. have not responded to our requests for data in digital form, so we scanned Figure 3 of [11] with the modern equipment available at the Lawrence Berkeley Laboratory and obtained numerical values in this way. The set of values is incomplete because in certain regions of that Figure there are so many points that it is impossible to separate them properly; there are however enough scanned points so that the conclusions below are independent of the remainder.

In Figure 1 we display the resulting values of $\log D_{LL}$ as a function of $\log D_{LLL}$ separately for each run, together with lines that correspond to $y = (2/3)x + a_1$, $y = 0.7x + a_2$, where x, y are the coordinates in those graphs and the values of a_1, a_2 are the same as in [11]. As one can see, the slope defined by the experimental points is almost exactly 0.7 for the experiment $C6$ ($Re = 6000$); it goes down as Re increases first to 18,000 and then to 300,000. The grid data, which we are told belong to a low Reynolds number, are particularly instructive: They follow the 2/3 line for a while, presumably while the separation r is in the dissipation scale, and then they bend towards the .7 line, as the separation emerges from the dissipation scale but the Reynolds number is not large enough to produce the asymptotic 2/3 scaling. The information at our disposal is not sufficient to estimate *a priori* where the bend should be. It is quite clear from these figures that one cannot view all these points as lying on a single line, and the data are compatible with incomplete similarity in r/Λ_K and an absence of similarity in Re, so that the exponent in the power law for D_{LL} and therefore the slopes of the lines in Figure 1 are slowly decreasing functions of Re when the separation r is in the inertial range.

To make this point another way, we display in Figure 2 the local slopes in these figures, defined as

$$s_i^{\text{local}} = \frac{y_i - y_1}{x_i - x_1}, \tag{3.1}$$

where (x_i, y_i) are the coordinates of the i-th point in the graph for specific experiment, and (x_1, y_1) is the first (leftmost) point in that graph. The local slopes that result from using successive points are too noisy for any conclusion to be drawn. Figure 2 clearly shows that the slopes decrease as Re increases for separations r outside the dissipation range. In particular, for the grid data (apparently lowest Re) the slope increases with the separation r as that separation emerges from the dissipation range.

4. Conclusion

Figures 1 and 2 show, we believe conclusively, that there is no reason to conclude with [11] that the Kolmogorov-Obukhov exponent is “definitely” different from 2/3 and independent of Re. The data as we presented them suggest to the contrary that the exponent slowly decreases with Re and tends to 2/3 as $Re \to \infty$. The experimental uncertainties detailed in [11], the uncertainty aboout the Reynolds number of the grid data, the small differences between the slopes under discussion, and the added uncertainties of the scanning, deter us from making the statement more emphatic yet.

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Figure Captions

Figure 1. Variation of $\log D_{LL}$ with $\log D_{LLL}$ plotted separately for the several values of Re:

a. Experiment C6 ($Re = 6000$); b. Experiment C18 ($Re = 18000$); c. Experiment J ($Re = 300000$); d. Grid turbulence.

Figure 2: Local slopes in Figure 1 for the several experiments. (Diamonds– C6; squares– C18; triangles– J; stars– grid turbulence). Note that the slope for grid turbulence start from 2/3 and increases, as one may expect from the calculation of the dissipation range (see the text).